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Strong Law of large number Law of the iterated logarithm for nonlinear probabilities

ZENGJING CHEN

SHANDONG UNIVERSITY

July 5, 2010



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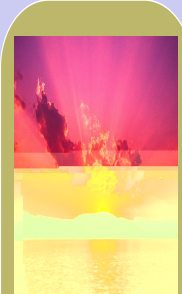
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Outline

- ◇ *History of LLN and LIL for probabilities*
- ◇ *Why to study LLN and LIL for capacities*
- ◇ *Nonlinear probabilities and nonlinear expectations*
- ◇ *Main results*
- ◇ *Applications*



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0.1. History of LLN and LIL for probability

Law of large number(LLN):

- (1) Brahmagupta (598-668), Cardano (1501-1576)
- (2) Jakob Bernoulli(1713), Poisson (1835)
- (3) Chebyshev, Markov, Borel(1909), Cantelli and Kolmogorov(IID).

Law of iterated logarithm(LIL):

- (1) Khintchine(1924) for Bernoulli model
Kolmogorov(1929), Hartman–Wintner(1941) (IID)
- (2) Levy(1937) for Martingale
- (3) Strassen(1964) for functional random variables.



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0.2. Strong LLN and LIL for probabilities

Assumption: $\{X_i\}$ IID , $S_n/n := \sum_{i=1}^n X_i$, $EX_1 = \mu$, Then

Theorem 1:Kolmogorov:

$$P\left(\lim_{n \rightarrow \infty} S_n/n = \mu\right) = 1$$

Theorem 2: Hartman–Wintner(1941): If $EX_1 = 0$, $EX_1^2 = \sigma^2$, Then

(a)

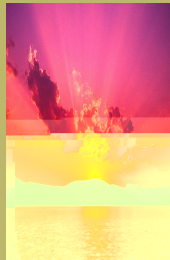
$$P\left(\limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \log \log n}} = \sigma\right) = 1$$

(b)

$$P\left(\liminf_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \log \log n}} = -\sigma\right) = 1$$

(c) Suppose that $C(\{x_n\})$ is the cluster set of a sequence of $\{x_n\}$ in R , then

$$P\left(C\left(\left\{ \frac{S_n}{\sqrt{2n \log \log n}} \right\}\right) = [-\sigma, \sigma]\right) = 1.$$



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0.3. Why to study LLN and LIL in Finance

THEOREM 1 (Black-Scholes, 1973:) *In complete markets, there exists a unique probability measure Q , such that the pricing of option at strike date T is given by $E_Q[e^{-rT}]$. Where $r = 0$ is interest rate of bond.*

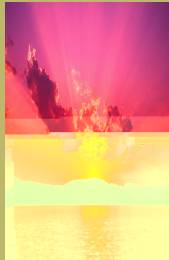
Monte Carlo, $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n X_i = E_Q[]$.

(Linear) expectation \leftarrow **Black-Scholes** \rightarrow Complete Markets

$\inf_{Q \in \mathcal{P}} E_Q[], \sup_{Q \in \mathcal{P}} E_Q[] \iff$ Incomplete Markets, Q is not unique, SET \mathcal{P} .

Super-pricing: $\inf_{Q \in \mathcal{P}} E_Q[], \sup_{Q \in \mathcal{P}} E_Q[]$. Nonlinear expectation!

$\lim_{n \rightarrow \infty} S_n/n = ?$



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0.4. Bernoulli Trials with ambiguity

Bernoulli Trials:

Repeated **independent** trials are called Bernoulli trials if there are only two possible outcomes for each trial and their probabilities **REMAIN** (are no longer) the same throughout the trials.

Let $X_i = 1$ if head occurs and $X_i = 0$ if tail occurs.

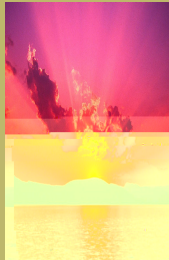
$$P(X_i = 1) = p, \quad P(X_i = 0) = 1 - p, \quad S_n := \sum_{i=1}^n X_i$$

If $p = 1/2$ (Unbalance), LLN stats

$$P\left(\lim_{n \rightarrow \infty} S_n/n = 1/2\right) = 1$$

Or

$$\lim_{n \rightarrow \infty} S_n/n = 1/2 \quad a.s. \quad (P)$$



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If a coin is balance. $P(X_i = 1) = \theta \in [1/3, 1/2]$.

Let $\mathcal{P} := \{P, \theta \in [1/3, 1/2]\}$.

$E_P[X_i] =$ Unknown,

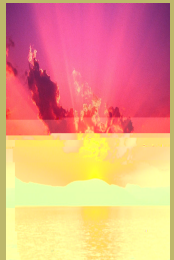
But $\max_{P \in \mathcal{P}} E_P[X_i] = 1/2, \quad \min_{P \in \mathcal{P}} E_P[X_i] = 1/3$.

Question: what is the limit $S_n/n \rightarrow?$

(a) Capacity: If $V(A) := \max_{P \in \mathcal{P}} P(A), \quad v(A) := \min_{P \in \mathcal{P}} P(A)$

Can S_n/n converge to $\max_{P \in \mathcal{P}} E_P[X_i]$ or $\min_{P \in \mathcal{P}} E_P[X_i]$ a.s. V or v ?

(b) The relation between the set of limit points of S_n/n and the interval of $\min_{P \in \mathcal{P}} E_P[X_i]$ and $\max_{P \in \mathcal{P}} E_P[X_i]$.



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0.5. Linear and Nonlinear Expectations

Kolmogorov: Linear expectation: $P : \mathcal{F} \rightarrow [0, 1], P(A) = E[I_A]$

$$P(A + B) = P(A) + P(B), A \cap B = \emptyset \Leftrightarrow E[\cdot + \cdot] = E[\cdot] + E[\cdot]$$

Expectation is a linear functional of random variable.

Nonlinear probability(capacity): $V(\cdot) : \mathcal{F} \rightarrow [0, 1]$ but

$$V(A + B) \neq V(A) + V(B), \text{ even } A \cap B = \emptyset.$$

Nonlinear expectation: $\mathbb{E}(\cdot)$ is nonlinear functional in the sense of

$$\mathbb{E}[\cdot + \cdot] \neq \mathbb{E}[\cdot] + \mathbb{E}[\cdot].$$

Capacity $V(A) = \mathbb{E}[I_A]$ is nonlinear.

Modes of nonlinear expectations and capacity

(1) Choquet expectations (Choquet 1953, physics)

$$C_V[X] := \int_0^\infty V(X \geq t) dt + \int_{-\infty}^0 [V(X \geq t) - 1] dt.$$

(2) g -expectation (Peng 1997)

(3) Sub-linear expectation (Peng 2007).

(a) Monotonicity: $X \geq Y$ implies $\mathbb{E}[X] \geq \mathbb{E}[Y]$.

(b) Constant preserving: $\mathbb{E}[c] = c, \forall c \in \mathbb{R}$.

(c) Sub-additivity: $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.

(d) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \forall \lambda \geq 0$.

(1) Distorted probability measure: $V(A) = g(P(A)), g : [0, 1] \rightarrow [0, 1]$.

(2) 2-alternating capacity: $V(A \cup B) \leq V(A) + V(B) - V(A \cap B)$

(3) $V(A) = \max_{P \in \mathcal{P}} P(A), \mathcal{P}$ set of Probability.

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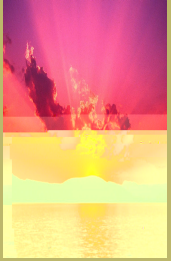
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3. Definition: capacity and nonlinear expectation

(1) Probability space : $(\Omega, \mathcal{F}, P) \Rightarrow (\Omega, \mathcal{F}, \mathcal{P})$. Where $\mathcal{P} := \{P : P \in \mathcal{P}\}$.

(2) Capacity: $P \Rightarrow (v, V)$, where

$$v(A) = \inf_{Q \in \mathcal{P}} Q(A), \quad V(A) = \sup_{Q \in \mathcal{P}} Q(A).$$

(3) Property:

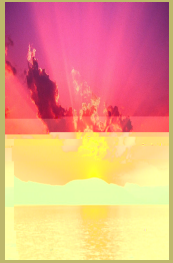
$$V(A) + V(A^c) \geq 1, \quad v(A) + v(A^c) \leq 1$$

but

$$V(A) + v(A^c) = 1.$$

(4) Nonlinear expectations: Lower-upper expectation $\mathcal{E}[\cdot]$ and $\mathbb{E}[\cdot]$

$$\mathcal{E}[\cdot] = \inf_{Q \in \mathcal{P}} E_Q[\cdot], \quad \mathbb{E}[\cdot] = \sup_{Q \in \mathcal{P}} E_Q[\cdot]$$



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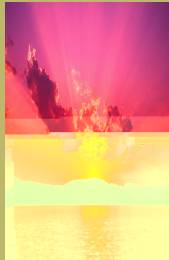
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$$V(AB) = V(A)V(B), v(AB) = v(A)v(B)$$

Theorem (Epstein, 02, Marinacci, 99, 05). Bounded, Polish, $C_V[X_i] = \underline{\mu}, C_V[X_i] = \bar{\mu}$. $\{X_i\}$ IID, then

$$v(\underline{\mu} \leq \liminf_{n \rightarrow \infty} S_n/n \leq \limsup_{n \rightarrow \infty} S_n/n \leq \bar{\mu}) = 1.$$

Where V is totally 2-alternating $V(A \cup B) \leq V(A) + V(B) - V(AB)$, here C_V and C_V is Choquet are integrals.

Note $C_V[X] \leq \mathcal{E}[X] \leq \mathbb{E}[X] \leq C_V[X], \forall X$.

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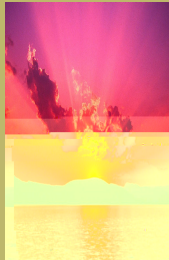
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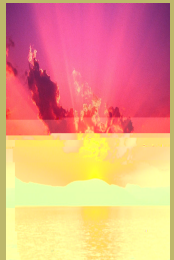
4.1. Limit theorem 1

Theorem: If $\{X_j\}$ is IID, then $\frac{S_n}{n}$ converges as $n \rightarrow \infty$ a.s. \forall if and only if

$$\mathcal{E}[X_1] = \mathbb{E}[X_1].$$

In this case,

$$\lim S_n/n = \mathcal{E}[X_1], \quad a.s. \quad \forall.$$



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5. Main results

THEOREM 3 $\{X_i\}_{i=1}^n$ IID under nonlinear expectation \mathbb{E} . Set $\bar{\mu} := \mathbb{E}[X_i]$, $\underline{\mu} := \mathcal{E}[X_i]$ and $S_n := \sum_{i=1}^n X_i$. If $\mathbb{E}[|X_i|^{1+\delta}] < \infty$ for $\delta > 0$. Then

(I)

$$V(\omega \in \Omega : \underline{\mu} \leq \liminf_{n \rightarrow \infty} S_n(\omega)/n \leq \limsup_{n \rightarrow \infty} S_n/n(\omega) \leq \bar{\mu}) = 1.$$

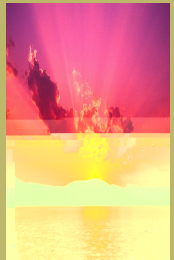
(II)

$$V(\omega \in \Omega : \limsup_{n \rightarrow \infty} S_n(\omega)/n = \bar{\mu}) = 1$$

$$V(\omega \in \Omega : \liminf_{n \rightarrow \infty} S_n(\omega)/n = \underline{\mu}) = 1.$$

(III) Suppose that $C(\{S_n(\omega)/n\})$ is the cluster set of a sequence of $\{S_n(\omega)/n\}$, then

$$V(\omega \in \Omega : C(\{S_n(\omega)/n\}) = [\underline{\mu}, \bar{\mu}]) = 1$$

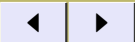
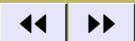


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6. Law of iterated logarithm for sub-linear expectations

THEOREM 4 $\{X_n\}$ bounded IID. $\mathbb{E}[X_1] = \mathcal{E}[X_1] = 0$, $\overline{\sigma}^2 := \mathbb{E}[X_1^2]$, $\underline{\sigma}^2 := \mathcal{E}[X_1^2]$. Let $S_n := \sum_{i=1}^n X_i$, $a_n := \sqrt{2n \lg \lg n}$, then

(I)

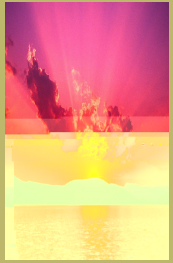
$$v\left(\underline{\sigma} \leq \limsup_n \frac{S_n}{a_n} \leq \overline{\sigma}\right) = 1;$$

(II)

$$v\left(\underline{\sigma} \leq \liminf_n \frac{S_n}{a_n} \leq \overline{\sigma}\right) = 1.$$

(III) Suppose that $C(\{x_n\})$ is the cluster set of a sequence of $\{x_n\}$ in \mathbb{R} , then

$$v\left(C(\{S_n/\sqrt{2n \log \log n}\}) \supset (\underline{\sigma}, \overline{\sigma})\right) = 1.$$



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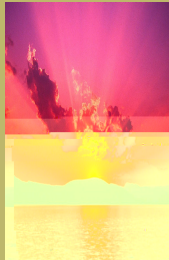
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7. Key of proof

THEOREM 5 Suppose X is distributed to G normal $N(0; [\sigma^2, -\sigma^2])$, where $0 < \sigma \leq \tau < \infty$. Let f be a bounded continuous function. Furthermore, if f is a positively even function, then, for any $b \in R$,

$$e^{-\frac{b^2}{2\sigma^2}} \mathcal{E}[f(X)] \leq \mathcal{E}[f(X - b)].$$



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8. Application

Total 100 balls in box, Black + Red + Yellow = 100,

Black = Red, Yellow $\in [30, 40]$, then $P_Y \in [3/10, 4/10]$.

Take a ball from this box,

$X_i = 1$, if ball is black, $X_i = 0$, if ball is Yellow, $X_i = -1$ for red.

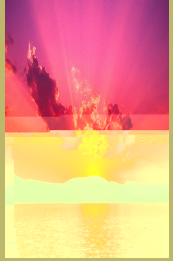
$S_n = \sum_{i=1}^n X_i$, is the excess frequency of black than Red

Then

(a) $\mathbb{E}[X_i] = \mathcal{E}[X_i] = 0$

(b)

$$\sqrt{6/10} \leq \limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \lg \lg n}} \leq \sqrt{7/10}.$$



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